A Formulation Procedure for No-Sparge and Batch-Sparge Recipes

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The object of mashing grain is to obtain sugar and other substances which ultimately become beer. This is done by combining the grain with hot water to encourage various chemical reactions, so that the desired extract becomes dissolved in the water and can then be run off into a collection vessel. It is common practice to try to maximize the extraction of goods from grain by not only draining the wort produced by the mash, but also by rinsing the grains with clear water to recover extract which otherwise would be absorbed by the grain and remain in the mash tun. Generally the clear water is added slowly and continuously while the wort is simultaneously drained from the mash tun. This often results in a runoff which is of a higher volume and lower specific gravity than the target recipe calls for. The target recipe is acheived by boiling off the excess volume which results in a higher specific gravity.

A simpler method of obtaining the extract is to simply drain the liquid from the mash tun without rinsing. This technique has become known as "no-sparge" brewing. Its chief disadvantage is that a significant amount of extract remains absorbed in the grain, which represents loss of extract efficiency. In a commercial setting this has obvious financial ramifications. Further, in order to compensate for the lost extract, extra grain is required, and so the issue of extra mash tun capacity must also be considered.

Economics aside, no-sparge brewing has been championed not only for its simplicity but also because the constant high gravity of the runoff inhibits the extraction of undesirable compounds from the grain, which can otherwise occur when the specific gravity of the wort in the mash tun falls below 1.019 (1). Anecdotal assertions of improved malt flavors (2) *may* be tied to the lower concentration of these undesirable compounds in a no-sparge wort as compared with traditionally-sparged wort, or higher concentrations of favorable compounds (3).

A sort of "hybrid" method involves performing a no-sparge runoff, followed by adding a charge of hot clear water to the drained mash. The "sparge" water is added batch-wise rather than being trickled onto the mash, and thus the method is often called "batch-sparge" brewing. The "sparge" water picks up much of the extract that was left behind from the first runoff, and is collected by performing a second runoff. The second runoff is typically of a specific gravity at or above the 1.019 "limit" and therefore any flavor benefits of no-sparge brewing would presumably apply to batch-sparge brewing as well (although a second factor, wort pH, must be considered as well). Because the "sparge" step recovers additional extract, the loss of efficiency compared to no-sparge brewing is generally less.

For both no-sparge and batch-sparge brewing, it is possible to obtain a smaller volume of higher-gravity wort compared to the target recipe, and therefore the brewer can use smaller vessels and heat sources, much like extract brewing. This is a distinct advantage for brewers who want to make all-grain recipes but who don't have the space or heating capacity generally required for fully-sparged worts. On the other hand, it is equally possible to run off a larger volume of lower-gravity wort, just as when continuous-sparge brewing. Thus, these methods offer increased kettle flexibility as an additional bonus.

In the literature on no-sparge brewing that has been published up till now, it has been made clear that extra grain is required in order to obtain the same volume and gravity as one would obtain if continuous-sparging. However, in most cases, figures like "25%" or "one-third" more grain are offered, without particular regard to whether these scale-ups will meet the final recipe requirements. In analyzing the recipe formulation math behind no-sparge and batch-saprge brewing, it is apparent that the extra grain required depends heavily on several factors, and can actually range from under 10% to 50% or more! Clearly more thought must go into recipe formulation if one is to expect a predictable outcome from a no-sparge or batch-sparge session.

Modelling the Mash

In order to quantify the process, we have to model what is happening in the mash tun. Here is a summary of the analysis (for the mathematical details, see the sidebar). Water is added to grain to produce wort through the usual chemical processes. In conventional continuous-sparge brewing, we collect as much of the potential extract as possible. This benchmarks our overall *efficiency* so that from a given quantity of grain, we can expect a certain total quantity of extract. If we assume that this extract is evenly distributed in the water, then if we simply run off the wort, we can expect the percentage of the total available extract collected to be equal to the percentage of the total liquid collected. And by knowing in advance how much liquid will be left behind, absorbed by the grain, we can figure the actual amount (volume) of runoff that will be collected. From this, we can "work backwards", starting with the desired runoff volume and gravity, and ending up with a specific amount of grain required to meet the recipe specifications.

For batch-sparge brewing, the process is then repeated, with the available extract for the second runoff being equal to whatever extract was left behind after the first runoff. The two runoffs are combined in the kettle to meet our desired volume and gravity requirements.

The equations that result are based on these assumptions:

- the extract is ***evenly distributed*** from one runoff to the next (in the case of batch-sparge brewing), so that the amount of extract recovered is proportional to the amount of liquid recovered;
- a traditional continuous-sparge session recovers ***all*** the extract produced by the mash. This isn't to say that the mash is assumed to be 100% efficient, but rather that whatever extract is created is fully recovered.

In reality these assumptions are probably not exactly accurate. For example, a brewer who assumes 75% extraction efficiency may in fact be obtaining 80% with regard to what is actually inside the mash tun, but is unable to get that last 5% into the kettle due to channelling or stagnant pockets of wort in the lautering process. Start with these equations and make adjustments based on your actual results.

(Mar02 update): You can account for this loss in the formulation if you know how much the loss actually is. In some cases, such as with false bottom mash tun designs, the loss can be signficant. In other cases, such as with a manifold design placed flush on the bottom of the mash tun with holes or slots facing only downward, the loss is probably negligible. Call this value "Vu". Set Vu = 0 to ignore its effects, or set it equal to the known value of the non-absorption system loss. Add the value of Vu to the desired kettle volume Vb and use the resulting value in place of Vb wherever it appears. If in doubt, use a value of Vu=0; you'll probably be close enough. The spreadsheet which accompanies this article allows for a value of Vu and automatically includes it in the calculations.

To formulate a no-sparge or batch-sparge recipe, you'll start with a "standard recipe" (the recipe you would use with traditional sparging techniques) which will then be scaled up by a factor of "S". The standard recipe will consist of a total grain bill weight "Wn", the resulting recipe volume of wort "Vr", and the resulting recipe original gravity "Gr". The standard recipe should be formulated assuming your normal, continuous-sparge brewing equipment and processes; in other words, it is what you would brew if you were going to continuous-sparge the mash.

Also, you'll need to pick a number representing the absorption of liquid by the grain. A well-drained mash will hold about 0.08 gal./lb ((0.67 l/kg), but you may want to consider using a larger figure so that you won't have to wait all day for that last drop of wort to find its way out of the mash tun. Also, depending on your mash tun design, you may not be able to recover all the liquid no matter how long you wait, and so a higher absorption value is required. A figure of 0.13 gal/lb (1.08 l/kg) or one pint per pound is a pretty good compromise between time saved and grain wasted. Call this factor "Ra".

Finally, decide on how much wort volume "Vb" you want to collect. This along with the absorption rate will determine how much liquid needs to be in the mash tun at runoff. This in turn determines the mash thickness "R". The mash thickness so determined may not, however, be the optimum figure for best mash chemistry performance, so I suggest you mash at a "normal" thickness less than or equal to "R", and add any additional water required to meet "R" just before recirculation and runoff. It is crucial that the mash be of thickness "R" just before runoff in order to meet your kettle volume and gravity requirements.

What you will determine is the following:

"S", the grain scale-up factor (the weight of each grain in the standard recipe is multiplied by "S" to obtain the no-sparge or batch-sparge recipe),
"Wg", the total weight of grain needed for the no-sparge or batch-sparge version,
"R", the required mash thickness at runoff,
"Vm", the total volume of mash water that has been added to achieve "R" at the first runoff,
"V1", the first runoff volume (equal to your "Vb" if no-sparge brewing),
"G1", the gravity of the first runoff,
"Vs", the required volume of "sparge" water (if batch-sparge brewing),
"V2", the second runoff volume (if batch-sparge brewing),
"G2", the second runoff gravity (if batch-sparge brewing),
"Vt", the total mash-tun capacity required to hold all the grain and water.

A note on "gravity" figures. Ideally you should use a gravity scale that is based on percentage of sugar, such as the Plato scale. You can use specific gravity "points", which are the digits following the decimal place in specific gravity (1.045 SG = 45 "points"), but there will be a *slight* error since SG does not correspond exactly linearly with sugar content. In any case, do NOT use specific gravity in the form 1.XXX, it will NOT work in these formulas!

You can use any units in these equations, as long as you use them throughout all the calculations. Please note that if you use gallons and pounds, the mash thickness "R" will also be in units of gallons per pound, not the more often used quarts per pound. Multiply gallons per pound by four to convert to quarts per pound if desired.

Grain occupies about 0.08 gal/lb (0.67 l/lb) when mixed with water. Call this figure "Q" and use it to find Vt, the total mash tun volume (capacity) required for the scaled-up recipe.

The Equations

For a no-sparge recipe:

```
S  = Vb / (Vb - (Ra x Wn))
G1 = Vr x Gr / Vb
R1 = Ra x S / (S - 1)
Wg = S x Wn
Vm = (Ra + R1) x Wg
Vt = Vm + (Q x Wg)
```

To minimize the extra grain required, collect as much wort as possible by establishing a thin mash just before runoff. Again, it's probably better to mash using a "normal" mash thickness, and thin the mash only after the conversion is complete.

When batch-sparge brewing, it turns out that the best extraction efficiency is obtained when the two runoffs are of equal volume (V1 = V2 = Vb/2):

```
R  = (Vb + SQRT{Vb^2 + (8 x Wn x Vb x Ra)}) / (4 x Wn)
S  = 1 / (1 - (Ra^2/R^2))
Wg = S x Wn
Vm = R x Wg
V1 = Vb /2
G1 = S x Vr x Gr / (V1 + (Ra x S x Wn))
Vs = V1
V2 = V1
G2 = Vr x Gr x (Ra/R) x (1 - Ra/R) / (Wn x (R - Ra))
Vt = Vm + (Q x Wg)
```

"SQRT{}" means take the square root of the expression inside the curly brackets {}

For a full explanation of the math behind these equations, see the sidebar.

Conclusion

No-sparge and batch-sparge brewing offer potential advantages in simplicity, flavor, and equipment, making them worthy additions to the brewer's aresenal of techniques. In the past, rule-of-thumb recipe scale-up guidelines offerred only approximate help in designing a no-sparge or batch-sparge version of a known recipe. By quantifying the changes required to predictably produce a no-sparge or batch-sparge wort, the brewer now has complete control over both the amount and the specific gravity of the wort produced, and the ability to optimize the process to minimize waste.

Sidebar: Derivation of the Formulas

The equations presented in the article were derived by starting with basic recipe formulation math, directed toward producing relationships describing the various parameters of a no-sparge or batch-sparge brew. Understanding the following is certainly not a requirement for using the formulas, but it offers backup to the validity of the formulas within the constraints of the assumptions outlined in the article.

Let's define a few relevant terms:

Definition of Variables Used in the Analysis -- Mash Conditions
Vr = recipe volume (can be more or less than available boiler capacity)
Gr = recipe gravity (expressed as "points/unit volume"; see below)
Vb = total volume to be sparged to boiler (there will be one runoff for no-sparge, two runoffs for batch-sparge)
E = standard continuous-sparge extraction efficiency (range 0 to 1, or percent ÷ 100)
Wn = weight of grain of the standard recipe
Wg = weight of grain of the no-sparge or batch-sparge recipe
Ra = absorption rate of the grain
R1 = final mash water volume to grain weight ratio (just before first runoff)
R2 = desired sparge water to grain ratio (just before second runoff, after sparge infusion and rest, batch-sparge only)
Ptn = total potential extract points of the standard recipe (see below)
Pt = total potential extract points of the no-sparge or batch-sparge recipe (see below)

Analyzing the Mash

It will be simpler to analyze a mashing session by assuming we've already scaled up our standard recipe to the no-sparge or batch-sparge version. Once we have a set of equations describing the mash conditions, we can "work backward" to derive the formulation process that got us here (that is, to find "S").

The total potential extract points value Pt is found by multiplying the weight of each grain in the recipe by its potential extract figure, and adding up the results. Typical figures are published in many places including the 1995 Special Issue of Zymurgy.

Pt = (weight of grain 1) x (potential extract for grain 1) +
(weight of grain 2) x (potential extract for grain 2) + ...

A continuous-sparge efficiency of E would then give us an available points figure Pa of:

Pa = Pt x E

Note the difference between "total points" and "gravity points". Total points refers to the total amount of sugar present in the volume, while gravity points refers to the concentration of sugar in a given volume of wort, in "points per (unit volume)". Gravity Points is equal to Total Points divided by the total volume of water in which those points are dissolved. It is a representation of the concentration of sugar, and typical units would be degrees Plato or just the decimal part of the specific gravity (e.g., 1.042 SG = 42 points per gallon). Note that the latter is not exactly accurate since specific gravity is not a linear representation of sugar concentration, though the error is small in the gravity ranges we're dealing with.

Using "points" to describe a quantity of sugar is convenient since extraction data is often given in terms of the specific gravity points
contributed by a certain weight of grain mashed in a certain volume of water. The unit of "points" allows us to express sugar content without having to actually figure out how many grams of sugar there are, and allows us to stay within the familiar confines of specific gravity or degrees Plato. It all comes out in the wash.

Now we can analyze the process to find all the other relevant factors:

Definition of Variables Used in the Analysis -- Recipe Results
Pn = predicted total extracted points for the standard recipe
Pa = predicted total extracted points for the no-sparge or batch-sparge recipe
Vm = total volume of mash water (at time of runoff)

Va = total volume of liquid absorbed by the grain
V1 = volume of first runoff (it's the only runoff if no-sparge)
G1 = gravity (points per unit volume) of first runoff
P1 = total points in first runoff
Pm = total points remaining in mash tun (in the liquid absorbed by the grain) after first runoff
Vs = volume of sparge water added to mash after first runoff (batch-sparge only)
V2 = volume of second runoff (batch-sparge only)
G2 = gravity of second runoff (batch-sparge only)
P2 = total points in second runoff (batch-sparge only)
Ps = total points remaining in mash tun (in the liquid absorbed by the grain) after second runoff (batch-sparge only)
Gb = gravity (points) of boiler contents after both runoffs (batch-sparge only)
Pb = total points in boiler kettle after both runoffs (batch-sparge only)

Just before recirculation and runoff, there will exist a certain water to grain ratio (mash thickness) R1 which is directly related to the total qantity of mash water used(Vm):

```
Vm = R1 x Wg
```

The gravity of the first runnings (in points per volume) is by definition the total actual points divided by the mash water volume:

```
G1 = Pa / Vm
```

The portion of the total mash water volume which is absorbed by the grain (more accurately, that which is left behind after the runoff) is

```
Va = Ra x Wg
```

For the first runoff volume V1, we will drain off all the liquid except that which is absorbed by the grain:

```
V1 = Vm - Va
   = (R1 x Wg) - (Ra x Wg)
   = (R1 - Ra) x Wg
```

Assuming sugar is evenly distributed throughout the wort, the first runoff will contain a proportion P1 of the total points Pa equal to the proportion of collected liquid to the total volume:

```
P1 / Pa = V1 / Vm

P1 = Pa x V1 / Vm
   = Pa x ((R1 - Ra) x Wg) / (R1 x Wg)
   = (1 - Ra / R1) x Pa
```

The number of points remaining in the mashtun after the first runoff is

```
Pm = Pa - P1
   = Pa - (1 - Ra / R1) x Pa
   = (Ra / R1 ) x Pa
```

If we were no-sparge brewing, we would quit here, leaving Pm points behind in the mashtun. Note, then, in order to obtain maximum no-sparge efficiency (that is, greatest P1 retrieved), we need to maximize R1 since both Pa and Ra are fixed constants. In other words, use as thin a mash (at the time of runoff) as is possible / practical, keeping in mind the volume of wort you want in the boiler. However, a thin mash may not be ideal for other reasons including efficiency of the mash chemistry. It's probably best to mash using "normal" values of R1 during the mash rest (0.33 gallons per pound, for example), and then thin out the mash to the target R1 just before recirculation & runoff, rather than mashing with a large value of R1.

We have seen that we will recover P1 points in the runoff. In a continuous-sparge session, we would have collected Pa points (this is how we defined Pa in the first place). Therefore the ratio of Pa to P1 is the same ratio as the extra grain required, so S can now be calculated:

```
S = Pa / P1
  = 1 / (1 - Ra / R1)
```

For the no-sparge recipe, increase the amount of each grain by multiplying by S. Note that this simultaneously increases the total available points Pa compared with the standard recipe points Pn:

```
Pa = S x Pn
```

so that the runoff gravity is

```
G1 = Pa / Vm
   = S x Pn / Vm
```

The original standard recipe will yield a volume of Vr at a gravity of Gr, from Pa total available points, with

```
Pn = Vr x Gr
```

and therefore

```
G1 = S x Vr x Gr / Vm
```

Note again that increasing R1 (a thinner mash) will decrease S (less extra grain required). The value of R1 is entirely up to the brewer.

No-Sparge Recipe Formulation (Design)

In the foregoing, we looked at how much wort is collected from a given amount of grain (and at what gravity) after specifying a mash thickness R1. While the volume and gravity obtained does yield the standard recipe volume Vr and gravity Gr after boiling or topping off, the boiler volume obtained is totally dependent on the value chosen for R1 (and Ra) and may be more or less than what we can or want to work with. A more practical problem then is to specify the boiler volume Vb and and gravity G1 of the runoff, and calculate the values of S and R1 required to acheive this volume. Note that the tradeoff here is efficiency (which is obtained by simply maximizing R1) versus predictability (specifying the target Vb).

Using the actual no-sparge or batch-sparge recipe points Pa, we find G1 (which for no-sparge is the same as Gb):

```
G1 = Pa / Vm
   = Pa / (Vb + Va)
```

From earlier we have `Pa = S x Pn` and `Va = Ra x Wg = Ra x S x Wn`, so

```
G1 = S x Pn / (Vb + Ra x S x Wn )
```

Rearranging,

```
S = Vb x G1 / (Pn - Ra x Wn x G1)
```

and since (in order to match the standard recipe) we need Pn = Vb x G1,

```
S = Pn / (Pn - Ra x Wn x G1)
```

Dividing through by G1,

```
S = Vb / (Vb - Ra x Wg)
```

Now that we know S we can solve the formula `S = 1 / (1 - Ra / R1)` for R1:

```
R1 = Ra x S / (S - 1)
```

Note here that you can end up with a wide range of values for R1 depending on your volume and gravity specifications. Generally, smaller values of Vb and higher values of Gb will result in smaller values of R1, and vice-versa. Remember that larger values of R1 can be handled by mashing with conventional water ratios and thinning out just before recirculation, rather than mashing thin.

At this point we have all we need to design a no-sparge recipe.

Batch-Sparge Recipe Analysis

When batch-sparging, the brewer performs the first runoff as described above, but continues by adding a quantity of hot sparge water to the mash tun, which will be stirred in and allowed to rest while it soaks more sugar out of the grain. As was the case with the initial mash, in this "second mash" a certain total sparge-water to grain ratio R2 exists, for a total liquid volume of R2 x Wg. Remember though that there is already liquid absorbed into the grain from the original mash, in the amount of Ra x Wg. So to get a mash thickness of R2 we need to add only the difference

```
Vs = R2 x Wg - Ra x Wg
   = (R2 - Ra) x Wg
```

The amount of sugar in the mash tun after the first runoff is Pm. Ideally, after the sparge infusion and a short rest, this sugar has been
redistributed evenly throughout the entire volume of liquid now present in the mash tun. We then collect the second runoff V2, which is the same volume as the added sparge water Vs since we don't need replace Va already in the mash tun:

```
V2 = Vs
```

This means that the initial mash volume accounts for both the first runoff and the absorbed wort, while the sparge infusion accounts only for the second runoff.

As with the first runoff, the number of points in V2 is the same as the proportion drained versus the total volume so

```
P2 / Pm = V2 / (V2 + Va)
P2 = Pm x Vs / (Vs + Va)
   = Pm x (R2 - Ra) x Wg / ((R2 - Ra) x Wg + Ra x Wg)
   = Pm x (1 - Ra / R2)
```

and the runoff gravity is

```
G2 = P2 / V2
```

The proportion of sugar left in the mash tun after sparging is the same proportion as the remaining (absorbed) liquid:

```
Ps / Pm = Ra / R2
Ps = Pm x (Ra / R2)
   = Pm x (Ra / R2) x (Ra / R1)
```

$= Pm \times (Ra^2 / (R1 \times R2))$

In the boiler, then, we now have a volume

$Vb = V1 + V2$

with a total number of points

$Pb = P1 + P2$

and therefore a gravity of

$Gb = Pb / Vb$

Optimizing the Batch-Sparge Process

Let's digress for a moment and see if we can optimize the extraction of sugar into the boiler for the batch-sparge case. The total number of points we put into the boiler was

$$
\begin{aligned}
Pb &= P1 + P2 \\
&= (1 - Ra / R1) \times Pa + (1 - Ra / R2) \times Pm \\
&= (1 - Ra / R1) \times Pa + (1 - Ra / R2) \times (Ra / R1) \times Pa \\
&= Pa \times [(1 - Ra / R1) + (1 - Ra / R2) \times (Ra / R1)] \\
&= Pa \times (1 - Ra^2 / (R1 \times R2))
\end{aligned}
$$

The ratio of Pb to Pa can be thought of as the "batch-sparge efficiency", or how many points make it to the boiler compared with all the points available. It is also our first look at a value of "S" for the batch-sparge process since we will need Pa/Pb times as much grain as the standard recipe calls for, so from the above relationship,

$$
\begin{aligned}
S &= Pa / Pb \\
&= 1 / (1 - Ra^2 / (R1 \times R2))
\end{aligned}
$$

Unfortunately at this point, we don't yet know the values of R1 and R2, so we can't evaluate S numerically. But we'll get there!

Let's take another look at

$Pb / Pa = 1 - (Ra^2 / (R1 \times R2))$

Since this represents the "batch-sparge efficiency", it is the expression we want to maximize. If you took calculus you might remember that to minimize or maximize a function, you must differentiate it with respect to the independent variable of interest, then set that expression equal to zero. However, we have two variables, R1 and R2. But, note that these variables are related in that we are obtaining a fixed volume of wort Vb, and therefore R2 depends on R1. Rewriting the expression for Vb in terms of R1 and R2, we see that

$$
\begin{aligned}
Vb &= V1 + V2 \\
&= (R1 - Ra) \times Wg + (R2 - Ra) \times Wg \\
&= (R1 + R2 - 2Ra) \times Wg
\end{aligned}
$$

Rearranging, we can write R2 in terms of R1:

$R2 = (Vb / Wg) + 2Ra - R1$

To maximize the batch-sparge efficiency expression, it is sufficient to minimize just the product R1 x R2. Rewriting R1 x R2 using the last equation,

$R1 \times R2 = R1 \times ((Vb / Wg) + 2Ra - R1)$

or

$R1 \times R2 = ((Vb / Wg) + 2Ra) \times R1 - R1^2$

We differentiate this expression with respect to R1 and set it equal to zero to maximize:

$0 = \boldsymbol{d}[((Vb / Wg) + 2Ra) \times R1 - R1^2] / \boldsymbol{d}R1$

$= (Vb / Wg) + 2Ra - 2R1$

Solving for R1

$R1 = (Vb / 2Wg) + Ra$

Rearranging we can write

$R1 - Ra = 1/2 \times Vb / Wg$

$(R1 - Ra) \times Wg = 1/2 \times Vb$

The term on the left is equal to V1, the first runoff volume. Therefore, we know that the optimum mash water ratio is that which yields half the boil volume on the first runoff. We then conclude that the optimum sparge water ratio must be identical to the mash water ratio in order to yield the other half of the boiler volume on the second runoff, and therefore

$R1 = R2$

is the condition for best efficiency with a specified boil volume Vb. As with no-sparge brewing, increasing R1 (and therefore R2) will also improve efficiency, but of course will result in a different boil volume. R1 and R2 must therefore be chosen to supply the desired boiler volume. In addition, for best efficiency, we want these two ratios to be equal. Let's see if we can tie these requirements together into an optimized formulation process.

Batch-Sparge Recipe Formulation (Design)

Again, the usual problem in the brewery is to determine the batch-sparge recipe based on the standard recipe, given the desired volume and gravity of the combined runoffs. We can achieve any specified runoff volume and gravity while at the same time optimizing the efficiency (that is, minimizing "S") by following the R1 = R2 rule. Since R1 = R2, let's just use "R" to stand for both ratios.

If we want half of the boiler volume in each of the runoffs, we need

$Vb/2 = Wg \times (R - Ra)$

Since $Wg = S \times Wn$ and $S = 1 / (1 - (Ra^2 / R2))$,

$$Vb/2 = S \times Wn \times (R - Ra)$$
$$= Wn \times (R - Ra) / (1 - (Ra^2 / R2))$$
$$= Wn \times (R - Ra) \times R^2 / (R^2 - Ra^2)$$

Rearranging,

$Vb/2 \times (R^2 - Ra^2) = Wn \times (R - Ra) \times R^2$

The difference of two squares on the left side is rewritten:

$Vb/2 \times (R - Ra) \times (R + Ra) = Wn \times (R - Ra) \times R^2$

Dividing through by (R - Ra),

$$Vb/2 \times (R + Ra) = Wn \times R^2$$

So

$$0 = Wn \times R^2 - Vb/2 \times R - Vb/2 \times Ra$$
$$= 2 \times Wn \times R^2 - Vb \times R - Vb \times Ra$$

which is a quadratic in R with two solutions

$$R = (Vb +/- SQRT\{Vb^2 + 8 \times Wn \times Vb \times Ra\}) / (4 \times Wn)$$

where everything inside the curly brackets after "SQRT" is considered to be under a square-root radical.

Note that the quantity under the radical is always greater than Vb. Since we need R > 0 (negative values for R don't make real-world sense!), the only root that "works" is

$$R = (Vb + SQRT\{Vb^2 + 8 \times Wn \times Vb \times Ra\}) / (4 \times Wn)$$

This gives us all we need to design an optimum batch-sparge session.

References

1. Bonham, Louis K., *No-Sparge Brewing -- An Old Technique Revisited*, The Experimental Brewer, Brewing Techniques Vol 6 Number 4, July/Aug 1998 (New Wine Press). See especially reference (4).

2. It seems that a post to the Homebrew Digest ("HBD") by George Fix (#977, 24SEP92) is considered the "definitive" introduction of the no-sparge technique to the general homebrewing public. In searching the HBD archives it appears to me that this post did not generate much discussion at the time. Not until #2196 (19SEP96) does a significant thread develop (started by Louis K. Bonham) , which lasts on and off till at least #2350 (15FEB97) with frequent references to the original Fix posting. There is much discussion of "enhanced maltiness" and reports of individual results in these Digests.
